

Braided monoidal categories and Doi Hopf modules for monoidal Hom-Hopf algebras

Shuangjian Guo¹, Xiaohui Zhang² and Shengxiang Wang^{3*}

1. School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guizhou 550025, China
2. Department of Mathematics, Southeast University, Nanjing 210096, China
3. School of Mathematics and Finance, Chuzhou University, Chuzhou 239000, China

Abstract We first introduce the notion of Doi Hom-Hopf modules and find the sufficient condition for the category of Doi Hom-Hopf modules to be monoidal. Also we obtain the condition for the monoidal Hom-algebra and monoidal Hom-coalgebra to be monoidal Hom-bialgebras. Second, we give the maps between the underlying monoidal Hom-Hopf algebras, Hom-comodule algebras and Hom-module coalgebras give rise to functors between the category of Doi Hom-Hopf modules and study tensor identities for monoidal categories of Doi Hom-Hopf modules. Furthermore, we construct a braiding on the category of Doi Hom-Hopf modules. Finally, as an application of our theory, we consider the braiding on the category of Hom-modules, the category of Hom-comodules and the category of Hom-Yetter-Drinfeld modules respectively.

Keywords Monoidal Hom-Hopf algebra; monoidal category; functor; Doi Hom-Hopf module.

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1 Introduction

The category ${}_A\mathcal{M}(H)^C$ of Doi-Hopf modules was introduced in [8], where H is a Hopf algebra, A a right H -comodule algebra and C a left H -module coalgebra. It is the category of the modules over the algebra A which are also comodules over the coalgebra C and satisfy certain compatibility condition involving H . The study of ${}_A\mathcal{M}(H)^C$ turned out to be very useful: it was shown in [8] that many categories such as the module and comodule categories over bialgebras, the Hopf modules category [20], and the Yetter-Drinfeld category [19] are special cases of ${}_A\mathcal{M}(H)^C$. For a further study of Doi-Hopf modules, we refer to [3], [4]. In [2], they proved that Yetter-Drinfeld modules are special cases of Doi-Hopf modules, therefore the category of Yetter-Drinfeld modules is a Grothendieck category.

*Correspondence: S X Wang: wangsx-math@163.com.

Hom-algebras and Hom-coalgebras were introduced by Makhlouf and Silvestrov in [16] as generalizations of ordinary algebras and coalgebras in the following sense: the associativity of the multiplication is replaced by the Hom-associativity and similar for Hom-coassociativity. They also defined the structures of Hom-bialgebras and Hom-Hopf algebras, and described some of their properties extending properties of ordinary bialgebras and Hopf algebras in [17] and [18]. Recently, many more properties and structures of Hom-Hopf algebras have been developed, see [5],[6], [7], [12], [14],[21],[22] and references cited therein.

Caenepeel and Goyvaerts [1] studied Hom-bialgebras and Hom-Hopf algebras from a categorical view point, and called them monoidal Hom-bialgebras and monoidal Hom-Hopf algebras respectively, which are slightly different from the above Hom-bialgebras and Hom-Hopf algebras. In [15], Makhlouf and Panaite defined Yetter-Drinfeld modules over Hom-bialgebras and shown that Yetter-Drinfeld modules over a Hom-bialgebra with bijective structure map provide solutions of the Hom-Yang-Baxter equation. Also Liu and Shen [14] studied Yetter-Drinfeld modules over monoidal Hom-bialgebras and called them Hom-Yetter-Drinfeld modules, and shown that the category of Hom-Yetter-Drinfeld modules is a braided monoidal categories. Chen and Zhang [7] defined the category of Hom-Yetter-Drinfeld modules in a slightly different way to [14], and shown that it is a full monoidal subcategory of the left center of left Hom-module category.

In this paper, we will discuss the following questions: how do we define a Doi Hopf module for monoidal Hom-Hopf algebras (we will call it a Doi Hom-Hopf module) such that the category of Doi Hom-Hopf modules is monoidal? In section 3, we will show that it is sufficient that (A, β) and (C, γ) are monoidal Hom-bialgebras with some extra conditons. As an example, we consider the category of Hom-Yetter-Drinfeld modules, which is well known to be a monoidal category from [14], this category is a special of our theory.

In section 4, we give the maps between the underlying monoidal Hom-Hopf algebras, Hom-comodule algebras and Hom-module coalgebras give rise to functors between the category of Doi Hom-Hopf modules. If the maps between the monodial Hom-algebras and monodial Hom-coalgebras are both monoidal Hom-bialgerbas maps, then we obtain functors monodial categories, and study tensor identities for monodial categories of Doi Hom-Hopf modules. As an application, we prove that the category of Doi Hom-Hopf modules has enough injective objects.

Suppose that we have a monoidal category of Doi Hom-Hopf modules. How do we define a braiding on this category? In section 5, we point out this comes down to giving a twisted convolution inverse map $\mathcal{Q} : C \otimes C \rightarrow A \otimes A$ satisfying the complicated compatibility conditions, as an application we consider the braiding on the category of Hom-modules, the category of Hom-comodules and the category of Hom-Yetter-Drinfeld modules respectively.

2 Preliminaries

Throughout this paper we work over a commutative ring k , we recall from [1] and [12] for some informations about Hom-structures which are needed in what follows.

Let \mathcal{C} be a category. We introduce a new category $\widetilde{\mathcal{H}}(\mathcal{C})$ as follows: objects are couples (M, μ) , with $M \in \mathcal{C}$ and $\mu \in \text{Aut}_{\mathcal{C}}(M)$. A morphism $f : (M, \mu) \rightarrow (N, \nu)$ is a morphism

$f : M \rightarrow N$ in \mathcal{C} such that $\nu \circ f = f \circ \mu$.

Let \mathcal{M}_k denotes the category of k -modules. $\mathcal{H}(\mathcal{M}_k)$ will be called the Hom-category associated to \mathcal{M}_k . If $(M, \mu) \in \mathcal{M}_k$, then $\mu : M \rightarrow M$ is obviously a morphism in $\mathcal{H}(\mathcal{M}_k)$. It is easy to show that $\mathcal{H}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (I, I), \tilde{a}, \tilde{l}, \tilde{r})$ is a monoidal category by Proposition 1.1 in [1]: the tensor product of (M, μ) and (N, ν) in $\mathcal{H}(\mathcal{M}_k)$ is given by the formula $(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$.

Assume that $(M, \mu), (N, \nu), (P, \pi) \in \mathcal{H}(\mathcal{M}_k)$. The associativity and unit constraints are given by the formulas

$$\begin{aligned}\tilde{a}_{M,N,P}((m \otimes n) \otimes p) &= \mu(m) \otimes (n \otimes \pi^{-1}(p)), \\ \tilde{l}_M(x \otimes m) &= \tilde{r}_M(m \otimes x) = x\mu(m).\end{aligned}$$

An algebra in $\mathcal{H}(\mathcal{M}_k)$ will be called a monoidal Hom-algebra.

Definition 2.1. A monoidal Hom-algebra is an object $(A, \alpha) \in \mathcal{H}(\mathcal{M}_k)$ together with a k -linear map $m_A : A \otimes A \rightarrow A$ and an element $1_A \in A$ such that

$$\begin{aligned}\alpha(ab) &= \alpha(a)\alpha(b); \quad \alpha(1_A) = 1_A, \\ \alpha(a)(bc) &= (ab)\alpha(c); \quad a1_A = 1_Aa = \alpha(a),\end{aligned}$$

for all $a, b, c \in A$. Here we use the notation $m_A(a \otimes b) = ab$.

Definition 2.2. A monoidal Hom-coalgebra is an object $(C, \gamma) \in \mathcal{H}(\mathcal{M}_k)$ together with k -linear maps $\Delta : C \rightarrow C \otimes C$, $\Delta(c) = c_{(1)} \otimes c_{(2)}$ (summation implicitly understood) and $\varepsilon : C \rightarrow k$ such that

$$\Delta(\gamma(c)) = \gamma(c_{(1)}) \otimes \gamma(c_{(2)}); \quad \varepsilon(\gamma(c)) = \varepsilon(c),$$

and

$$\gamma^{-1}(c_{(1)}) \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes \gamma^{-1}(c_{(2)}), \quad \varepsilon(c_{(1)})c_{(2)} = \varepsilon(c_{(2)})c_{(1)} = \gamma^{-1}(c)$$

for all $c \in C$.

Definition 2.3. A monoidal Hom-bialgebra $H = (H, \alpha, m, \eta, \Delta, \varepsilon)$ is a bialgebra in the symmetric monoidal category $\mathcal{H}(\mathcal{M}_k)$. This means that (H, α, m, η) is a monoidal Hom-algebra, $(H, \alpha, \Delta, \varepsilon)$ is a monoidal Hom-coalgebra and that Δ and ε are morphisms of Hom-algebras, that is,

$$\begin{aligned}\Delta(ab) &= a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}; \quad \Delta(1_H) = 1_H \otimes 1_H, \\ \varepsilon(ab) &= \varepsilon(a)\varepsilon(b), \quad \varepsilon(1_H) = 1_H.\end{aligned}$$

Definition 2.4. A monoidal Hom-Hopf algebra is a monoidal Hom-bialgebra (H, α) together with a linear map $S : H \rightarrow H$ in $\mathcal{H}(\mathcal{M}_k)$ such that

$$S * I = I * S = \eta\varepsilon, \quad S\alpha = \alpha S.$$

Definition 2.5. Let (A, α) be a monoidal Hom-algebra. A right (A, α) -Hom-module is an object $(M, \mu) \in \mathcal{H}(\mathcal{M}_k)$ consists of a k -module and a linear map $\mu : M \rightarrow M$ together with a morphism $\psi : M \otimes A \rightarrow M$, $\psi(m \cdot a) = m \cdot a$, in $\mathcal{H}(\mathcal{M}_k)$ such that

$$(m \cdot a) \cdot \alpha(b) = \mu(m) \cdot (ab); \quad m \cdot 1_A = \mu(m),$$

for all $a \in A$ and $m \in M$. The fact that $\psi \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ means that

$$\mu(m \cdot a) = \mu(m) \cdot \alpha(a).$$

A morphism $f : (M, \mu) \rightarrow (N, \nu)$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called right A -linear if it preserves the A -action, that is, $f(m \cdot a) = f(m) \cdot a$. $\widetilde{\mathcal{H}}(\mathcal{M}_k)_A$ will denote the category of right (A, α) -Hom-modules and A -linear morphisms.

Definition 2.6. Let (C, γ) be a monoidal Hom-coalgebra. A right (C, γ) -Hom-comodule is an object $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k -linear map $\rho_M : M \rightarrow M \otimes C$ notation $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ such that

$$m_{[0][0]} \otimes (m_{[0][1]} \otimes \gamma^{-1}(m_{[1]})) = \mu^{-1}(m_{[0]}) \otimes \Delta_C(m_{[1]}); \quad m_{[0]} \varepsilon(m_{[1]}) = \mu^{-1}(m),$$

for all $m \in M$. The fact that $\rho_M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ means that

$$\rho_M(\mu(m)) = \mu(m_{[0]}) \otimes \gamma(m_{[1]}).$$

Morphisms of right (C, γ) -Hom-comodule are defined in the obvious way. The category of right (C, γ) -Hom-comodules will be denoted by $\widetilde{\mathcal{H}}(\mathcal{M}_k)^C$.

Definition 2.7. Let (H, α) be a monoidal Hom-Hopf algebra. A monoidal Hom-algebra (A, β) is called a right (H, α) -Hom-comodule algebra if (A, β) is a right (H, α) -Hom-comodule with coaction $\rho_A : A \rightarrow A \otimes H$, $\rho_A(a) = a_{[0]} \otimes a_{[1]}$ such that the following conditions satisfy:

$$\rho_A(ab) = a_{[0]} b_{[0]} \otimes a_{[1]} b_{[1]}, \quad \rho_A(1_A) = 1_A \otimes 1_H,$$

for all $a, b \in A$.

3 Making the category of Doi Hom-Hopf modules into a monoidal category

Definition 3.1. Let (H, α) be a monoidal Hom-Hopf algebra. A monoidal Hom-coalgebra (C, γ) is called a left (H, α) -Hom-module coalgebra, if (C, γ) is a left (H, α) -Hom-module with action $\phi : H \otimes C \rightarrow C$, $\phi(h \otimes c) = h \cdot c$ such that the following conditions hold:

$$\begin{aligned} \Delta_C(h \cdot c) &= h_{(1)} \cdot c_{(1)} \otimes h_{(2)} \cdot c_{(2)}, \\ \varepsilon_C(h \cdot c) &= \varepsilon_C(c) \varepsilon_H(h), \end{aligned}$$

for all $c \in C$ and $g, h \in H$.

A Doi Hom-Hopf datum is a triple (H, A, C) , where H is a monoidal Hom-Hopf algebra, A a right (H, α) -Hom comodule algebra and (C, γ) a left (H, α) -Hom module coalgebra.

Definition 3.2. Given a Doi Hom-Hopf datum (H, A, C) . A Doi Hom-Hopf module (M, μ) is a left (A, β) -Hom-module which is also a right (C, γ) -Hom-comodule with the

coaction structure $\rho_M : M \rightarrow M \otimes C$ defined by $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ such that the following compatible condition holds: for all $m \in M$ and $a \in A$,

$$\rho_M(a \cdot m) = a_{[0]} \cdot m_{[0]} \otimes a_{[1]} \cdot m_{[1]}.$$

A morphism between left-right Doi Hom-Hopf modules is a k -linear map which is a morphism in the categories ${}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)$ and $\widetilde{\mathcal{C}}(\mathcal{M}_k)^C$ at the same time. ${}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$ will denote the category of left-right Doi Hom-Hopf modules and morphisms between them.

Now suppose that C and A are both monoidal Hom-bialgebras.

Proposition 3.3. Let $(M, \mu) \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$, $(N, \nu) \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$. Then we have $M \otimes N \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$ with structures:

$$\begin{aligned} a \cdot (m \otimes n) &= a_{(1)} \cdot m \otimes a_{(2)} \cdot n, \\ \rho_{M \otimes N}(m \otimes n) &= m_{[0]} \otimes n_{[0]} \otimes m_{[1]} n_{[1]} \end{aligned}$$

if and only if the following condition holds:

$$a_{(1)[0]} \otimes a_{(2)[0]} \otimes (a_{(1)[1]} \cdot c)(a_{(2)[1]} \cdot d) = a_{[0](1)} \otimes a_{[0](2)} \otimes a_{[1]} \circ (cd), \quad (3.1)$$

for all $a \in A$ and $c, d \in C$. Furthermore, the category $\mathcal{C} = {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$ is a monoidal category.

Proof. It is easy to see that $M \otimes N$ is a left (A, β) -module and that $M \otimes N$ is a right (C, γ) -comodule. Now we check that the compatibility condition holds:

$$\begin{aligned} &\rho_{M \otimes N}(a \cdot (m \otimes n)) \\ &= (a_{(1)} \cdot m)_{[0]} \otimes (a_{(2)} \cdot n)_{[0]} \otimes (a_{(1)} \cdot m)_{[1]} (a_{(2)} \cdot n)_{[1]} \\ &= a_{(1)[0]} \cdot m_{[0]} \otimes a_{(2)[0]} \cdot n_{[0]} \otimes (a_{(1)[1]} \cdot m_{[1]})(a_{(2)[1]} \cdot n_{[1]}) \\ &\stackrel{(3.1)}{=} a_{[0](1)} \cdot m_{[0]} \otimes (a_{[0](2)} \cdot n_{[0]}) \otimes a_{[1]} \cdot (m_{[1]} n_{[1]}) \\ &= a_{[0]} \cdot (m_{[0]} \otimes n_{[0]}) \otimes a_{[1]} \cdot (m_{[1]} n_{[1]}). \end{aligned}$$

So $M \otimes N \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$.

Conversely, one can easily check that $A \otimes C \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$, let $m = 1 \otimes c$ and $n = 1 \otimes d$ for any $c, d \in C$ and easily get Eq.(3.2).

Furthermore, k is an object in ${}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$ with structures:

$$a \cdot x = \varepsilon_A(a)x, \quad \rho(x) = x \otimes 1_C,$$

for all $x \in k$ if and only if the following condition holds:

$$\varepsilon_A(a)1_C = \varepsilon_A(a_{(0)})(a_{(1)} \cdot 1_C), \quad (3.2)$$

for all $a \in A$. Then it is easy to get that $(\mathcal{C} = {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C, \otimes, k, a, l, r)$ is a monoidal category, where a, l, r are given by the formulas:

$$\begin{aligned} \tilde{a}_{M,N,P}((m \otimes n) \otimes p) &= \mu(m) \otimes (n \otimes \pi^{-1}(p)), \\ \tilde{l}_M(x \otimes m) &= \tilde{r}_M(m \otimes x) = x\mu(m), \end{aligned}$$

for $(M, \mu), (N, \nu), (P, \pi) \in \mathcal{C}$. \square

We call $G = (H, A, C)$ a *monoidal Doi Hom-Hopf datum* if G is a Doi Hom-Hopf datum, and A, C are Hom-bialgebras with the additional compatibility relations Eq.(3.1) and Eq.(3.2).

We will give an example for the monoidal category ${}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$. First, we give the definition of Yetter-Drinfeld modules over a monoidal Hom-Hopf algebra, which is also introduced by Liu and Shen in [14] similarly.

Definition 3.4. Let (H, α) be a monoidal Hom-Hopf algebra. A left-right (H, α) -Hom-Yetter-Drinfeld module is an object (M, μ) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, such that (M, μ) a left (H, α) -Hom-module and a right (H, α) -Hom-comodule with the following compatibility condition:

$$h_{(1)} \cdot m_{[0]} \otimes h_{(2)} m_{[1]} = \mu((h_{(2)} \cdot \mu^{-1}(m))_{[0]}) \otimes (h_{(2)} \cdot \mu^{-1}(m))_{[1]} h_{(1)} \quad (3.3)$$

for all $h \in H$ and $m \in M$. We denote by ${}_H\mathcal{HYD}^H$ the category of left-right (H, α) -Hom-Yetter-Drinfeld modules, morphisms being left (H, α) -linear right (H, α) -colinear maps.

Proposition 3.5. One has that Eq. (3.3) is equivalent to the following equation:

$$\rho(h \cdot m) = \alpha(h_{(2)(1)}) \cdot m_{[0]} \otimes (h_{(2)(2)} \alpha^{-1}(m_{[1]})) S^{-1}(h_{(1)}), \quad (3.4)$$

for all $h \in H$ and $m \in M$.

Proof. For one thing, we compute

$$\begin{aligned} & \mu((h_{(2)} \cdot \mu^{-1}(m))_{[0]}) \otimes ((h_{(2)} \cdot \mu^{-1}(m))_{[1]}) h_{(1)} \\ \stackrel{(3.4)}{=} & \mu(\alpha(h_{(2)(2)(1)}) \cdot \mu^{-1}(m_{[0]})) \otimes ((h_{(2)(2)(2)} \alpha^{-2}(m_{[1]})) S^{-1}(h_{(2)(1)})) h_{(1)} \\ = & \alpha(h_{(2)(1)}) \cdot m_{[0]} \otimes (h_{(2)(2)} \alpha^{-1}(m_{[1]})) (S^{-1}(h_{(1)(2)}) h_{(1)(1)}) \\ = & h_{(1)} \cdot m_{[0]} \otimes h_{(2)} m_{[1]}. \end{aligned}$$

For another, we have

$$\begin{aligned} & h_{(2)(1)} \cdot m_{[0]} \otimes (h_{(2)(2)} m_{[1]}) S^{-1}(h_{(1)}) \\ \stackrel{(3.3)}{=} & \mu((h_{(2)(2)} \cdot \mu^{-1}(m))_{[0]}) \otimes ((h_{(2)(2)} \cdot \mu^{-1}(m))_{[1]}) h_{(2)(1)} S^{-1}(h_{(1)}) \\ = & \mu((\alpha^{-1}(h_{(2)}) \cdot \mu^{-1}(m))_{[0]}) \otimes \alpha((\alpha^{-1}(h_{(2)}) \cdot \mu^{-1}(m))_{[1]}) (h_{(1)(2)} S^{-1}(h_{(1)(1)})) \\ = & \mu((\alpha^{-2}(h) \cdot \mu^{-1}(m))_{[0]}) \otimes \alpha^2((\alpha^{-2}(h) \cdot \mu^{-1}(m))_{[1]}) \\ = & (\alpha^{-1}(h) \cdot m)_{[0]} \otimes \alpha((\alpha^{-1}(h) \cdot m)_{[1]}), \end{aligned}$$

which implies Eq. (3.4). \square

Theorem 3.6. Let (H, α) be a monoidal Hom-Hopf algebra with a bijective antipode.

(1) H can be made into a right $H^{op} \otimes H$ -Hom-comodule algebra. The coaction $H \rightarrow H \otimes H^{op} \otimes H$ is given by the formula:

$$h \mapsto \alpha(h_{(2)(1)}) \otimes (S^{-1}(\alpha^{-1}(h_{(1)})) \otimes h_{(2)(2)}).$$

(2) H can be made into a left $H^{op} \otimes H$ -Hom module coalgebra. The action of $H^{op} \otimes H$ on H is given by the formula:

$$(h \otimes k) \triangleright c = (k \alpha^{-1}(c)) \alpha(h).$$

(3) The category ${}_H\mathcal{HYD}^H$ of left-right Hom-Yetter-Drinfeld modules is isomorphic to a category of Doi Hom-Hopf modules, namely ${}_H\widetilde{\mathcal{H}}(\mathcal{M}_k)(H^{op} \otimes H)^H$.

Proof. (1) We first prove that H is a right $H^{op} \otimes H$ -Hom comodule. For any $h \in H$,

$$\begin{aligned}
& (\alpha^{-1} \otimes \Delta_{H^{op} \otimes H})\rho_H(h) \\
&= h_{(2)(1)} \otimes \Delta_{H^{op} \otimes H}(S^{-1}(\alpha^{-1}(h_{(1)})) \otimes h_{(2)(2)}) \\
&= h_{(2)(1)} \otimes S^{-1}(\alpha^{-1}(h_{(1)(2)})) \otimes h_{(2)(2)(1)} \otimes S^{-1}(\alpha^{-1}(h_{(1)(1)})) \otimes h_{(2)(2)(2)} \\
&= \alpha(h_{(2)(1)(1)}) \otimes S^{-1}(\alpha^{-1}(h_{(1)(2)})) \otimes h_{(2)(1)(2)} \otimes S^{-1}(\alpha^{-1}(h_{(1)(1)})) \otimes \alpha^{-1}(h_{(2)(2)}) \\
&= \alpha^2(h_{(2)(2)(1)(1)}) \otimes S^{-1}(\alpha^{-1}(h_{(2)(1)})) \otimes \alpha(h_{(2)(2)(1)(2)}) \otimes S^{-1}(\alpha^{-2}(h_{(1)})) \otimes h_{(2)(2)(2)} \\
&= \alpha^2(h_{(2)(1)(2)(1)}) \otimes S^{-1}(h_{(2)(1)(1)}) \otimes \alpha(h_{(2)(1)(1)(2)}) \otimes S^{-1}(\alpha^{-2}(h_{(1)})) \otimes \alpha^{-1}(h_{(2)(2)}) \\
&= \rho(\alpha(h_{(2)(1)})) \otimes S^{-1}(\alpha^{-2}(h_{(1)})) \otimes \alpha^{-1}(h_{(2)(2)}) \\
&= (\rho_H \otimes \alpha^{-1})\rho_H(h)
\end{aligned}$$

So H is a right $H^{op} \otimes H$ -Hom comodule. We also have

$$\begin{aligned}
& \rho(hg) = \alpha(h_{(2)(1)}g_{(2)(1)}) \otimes (S^{-1}(h_{(1)}g_{(1)}) \otimes \alpha^{-1}(h_{(2)(2)}g_{(2)(2)})) \\
&= \alpha(h_{(2)(1)})\alpha(g_{(2)(1)}) \otimes (S^{-1}(h_{(1)})S^{-1}(g_{(1)}) \otimes \alpha^{-1}(h_{(2)(2)})\alpha^{-1}(g_{(2)(2)})) \\
&= (\alpha(h_{(2)(1)}) \otimes (S^{-1}(h_{(1)}) \otimes \alpha^{-1}(h_{(2)(2)})))(\alpha(g_{(2)(1)}) \otimes (S^{-1}(g_{(1)}) \otimes \alpha^{-1}(g_{(2)(2)}))) \\
&= \rho_H(h)\rho_H(g).
\end{aligned}$$

(2) Now we prove that H is a $H^{op} \otimes H$ -Hom comodule. For any $h, l, k, m, c \in H$, we have

$$\begin{aligned}
& (\alpha(l) \otimes \alpha(m)) \triangleright [(h \otimes k) \triangleright c] \\
&= (\alpha(l) \otimes \alpha(m)) \triangleright (k\alpha^{-1}(c))\alpha(h) \\
&= [\alpha(m)[(\alpha^{-1}(k)\alpha^{-2}(c))h]]\alpha^2(l) \\
&= [\alpha(m)[k(\alpha^{-2}(c))\alpha^{-1}(h)]]\alpha^2(l) \\
&= \alpha(mk)[[\alpha^{-1}(c))h]\alpha(l)] \\
&= \alpha(mk)[c(hl)] = mk[c\alpha(hl)] \\
&= (hl \otimes mk) \triangleright \alpha(c) = [(l \otimes m)(h \otimes k)] \triangleright \alpha(c),
\end{aligned}$$

and this implies that H is a $H^{op} \otimes H$ -Hom module.

Using the fact that (H, α) is an (H, α) -Hom-bimodule algebra, we can obtain that (H, α) is a left $H^{op} \otimes H$ -Hom-module coalgebra.

(3) Let (M, \cdot) be a left (H, α) -module and (M, ρ_M) a right (H, α) -comodule. Then $M \in {}_H\widetilde{\mathcal{H}}(\mathcal{M}_k)(H^{op} \otimes H)^H$ if and only if

$$\begin{aligned}
\rho_M(h \cdot m) &= \alpha(h_{(2)(1)}) \cdot m_{[0]} \otimes (S^{-1}(\alpha^{-1}(h_{(1)})) \otimes h_{(2)(2)}) \triangleright m_{[1]} \\
&= \alpha(h_{(2)(1)}) \cdot m_{[0]} \otimes (h_{(2)(2)}\alpha^{-1}(m_{[1]}))S^{-1}(h_{(1)}),
\end{aligned}$$

for all $h \in H$ and $m \in M$. Thus ${}_H\widetilde{\mathcal{H}}(\mathcal{M}_k)(H^{op} \otimes H)^H$ is isomorphic to ${}_H\mathcal{HYD}^H$. \square

Example 3.7. Let (H, α) be a monoidal Hom-Hopf algebra, we have shown that the category of Doi Hom-Hopf modules ${}_H\widetilde{\mathcal{H}}(\mathcal{M}_k)(H^{op} \otimes H)^H$ and the category of Hom-Yetter-Drinfeld modules ${}_H\mathcal{HYD}^H$ are isomorphic. Recall from that [14] that the category of

Hom-Yetter-Drinfeld modules is a monoidal category; let us check that it is a special case of Proposition 3.3. Indeed, take $A = H$ and $C = H^{op}$ as monoidal Hom-bialgebras. Let $a = h, c = k$ and $d = g$. Then the left-hand side amounts to

$$\begin{aligned} & h_{[0](1)} \otimes h_{[0](2)} \otimes h_{[1]} \cdot (k \bullet g) \\ &= \alpha(h_{(2)(1)(1)}) \otimes \alpha(h_{(2)(1)(2)}) \otimes (S^{-1}(\alpha^{-1}(h_{(1)})) \otimes h_{(2)(2)}) \cdot (gk) \\ &= \alpha(h_{(2)(1)(1)}) \otimes \alpha(h_{(2)(1)(2)}) \otimes [h_{(2)(2)}\alpha^{-1}(gk)]S^{-1}(h_{(1)}). \end{aligned}$$

The right-hand side is

$$\begin{aligned} & h_{(1)[0]} \otimes h_{(2)[0]} \otimes (h_{1} \cdot k)(h_{[1](2)} \cdot g) \\ &= \alpha(h_{(1)(2)(1)}) \otimes \alpha(h_{(2)(2)(1)}) \otimes ((S^{-1}(\alpha^{-1}(h_{(1)(1)})) \otimes h_{(2)(2)(1)}) \cdot k) \\ & \quad \bullet ((S^{-1}(\alpha^{-1}(h_{(1)(2)})) \otimes h_{(2)(2)(2)}) \cdot g) \\ &= \alpha(h_{(1)(2)(1)}) \otimes \alpha(h_{(2)(2)(1)}) \otimes ((h_{(2)(2)(2)}\alpha^{-1}(g))S^{-1}(h_{(1)(2)})) \\ & \quad ((h_{(2)(2)(1)}\alpha^{-1}(k))S^{-1}(h_{(1)(1)})) \\ &= \alpha(h_{(1)(2)(1)}) \otimes \alpha(h_{(2)(2)(1)}) \otimes ((h_{(2)(2)(2)}\alpha^{-1}(g))[S^{-1}(\alpha^{-1}(h_{(1)(2)}))h_{(2)(2)(1)}]) \\ & \quad kS^{-1}(h_{(1)(1)}) \\ &= \alpha(h_{(1)(1)(2)}) \otimes \alpha(h_{(2)(1)(2)}) \otimes ((\alpha^{-1}(h_{(2)(2)})\alpha^{-1}(g))[S^{-1}(h_{(1)(1)(2)})\alpha^{-1}(h_{(2)(1)})]) \\ & \quad kS^{-1}(\alpha(h_{(1)(1)(1)})) \\ &= \alpha(h_{(2)(1)(1)}) \otimes \alpha(h_{(2)(1)(2)}) \otimes ((h_{(2)(2)(2)}\alpha^{-1}(g))[S^{-1}(h_{(2)(1)(1)})h_{(2)(1)(1)}]) \\ & \quad kS^{-1}(\alpha^{-1}(h_{(1)})) \\ &= \alpha(h_{(2)(1)(1)}) \otimes \alpha(h_{(2)(1)(2)}) \otimes ((h_{(2)(2)}g)kS^{-1}(\alpha^{-1}(h_{(1)}))) \\ &= \alpha(h_{(2)(1)(1)}) \otimes \alpha(h_{(2)(1)(2)}) \otimes [(\alpha^{-1}(h_{(2)(2)})\alpha^{-1}(g))k]S^{-1}(h_{(1)}) \\ &= \alpha(h_{(2)(1)(1)}) \otimes \alpha(h_{(2)(1)(2)}) \otimes [h_{(2)(2)}\alpha^{-1}(gk)]S^{-1}(h_{(1)}). \end{aligned}$$

□

4 Tensor identities

Theorem 4.1. Given two Doi Hom-Hopf datums (H, A, C) and (H', A', C') , suppose that a morphism $\xi : (H, A, C) \rightarrow (H', A', C')$ consist three maps $\varphi : H \rightarrow H'$, $\psi : A \rightarrow A'$ and $\phi : C \rightarrow C'$ which are respectively monoidal Hom-Hopf algebra, Hom-algebra and Hom-coalgebra maps satisfying

$$\phi(h \cdot c) = \varphi(h) \cdot \phi(c), \quad (4.1)$$

$$\rho_A(\psi(a)) = \psi(a_{[0]}) \otimes \varphi(a_{[1]}), \quad (4.2)$$

for all $c \in C$, $h \in H$ and $a \in A$. Then we have a functor $F : {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C \rightarrow {}_{A'}\widetilde{\mathcal{H}}(\mathcal{M}_k)(H')^{C'}$, defined as follows:

$$F(M) = A' \otimes_A M,$$

where (A', β') is a left (A, β) -module via ψ and with structure maps defined by

$$b' \cdot (a' \otimes_A m) = \beta'^{-1}(b')a' \otimes_A \mu(m), \quad (4.3)$$

$$\rho_{F(M)}(a' \otimes_A m) = a'_{[0]} \otimes_A m_{[0]} \otimes a'_{[1]} \cdot \phi(m_{[1]}), \quad (4.4)$$

for all $a', b' \in A'$ and $m \in M$.

Proof. Let us show that $A' \otimes_A M$ is an object of ${}_{A'}\widetilde{\mathcal{H}}(\mathcal{M}_k)(H')^{C'}$. It is routine to check that $F(M)$ is a left (A', β') -module. For this, we need to show that $A' \otimes_A M$ is a right (C', γ') -comodule and satisfy the compatible condition, for any $m \in M$ and $a', b' \in A'$, we have

$$\begin{aligned} \rho_{F(M)}(b' \cdot (a' \otimes_A m)) &= \rho_{F(M)}(\beta'^{-1}(b')a' \otimes_A \mu(m)) \\ &= \beta'^{-1}(b'_{[0]})a'_{[0]} \otimes_A \mu(m_{[0]}) \otimes [\beta'^{-1}(b'_{[1]})a'_{[1]}] \cdot \phi(\gamma(m_{[1]})) \\ &= b'_{[0]}[a'_{[0]} \otimes_A m_{[0]}] \otimes b'_{[1]}[a'_{[1]} \cdot \phi(m_{[1]})] \\ &= b' \cdot (a'_{[0]} \otimes_A m_{[0]} \otimes a'_{[1]} \cdot \phi(m_{[1]})) = b' \rho_{F(M)}(a' \otimes_A m) \end{aligned}$$

i.e., the compatible condition holds. It remains to prove that $A' \otimes_A M$ is a right (C', γ') -comodule. For any $m \in M$ and $a' \in A'$, we have

$$\begin{aligned} &(\rho_{F(M)} \otimes id_{C'})\rho_{F(M)}(a' \otimes_A m) \\ &= (\rho_{F(M)} \otimes id_{C'})(a'_{[0]} \otimes_A m_{[0]} \otimes a'_{[1]} \cdot \phi(m_{[1]})) \\ &= [a'_{[0][0]} \otimes_A m_{[0][0]} \otimes a'_{[0][1]} \cdot \phi(m_{[0][1]})] \otimes a'_{[1]} \cdot \phi(m_{[1]}) \\ &= [\beta'^{-1}(a'_{[0]}) \otimes_A \mu^{-1}(m_{[0]}) \otimes a'_{1} \cdot \phi(m_{1})] \otimes \alpha'(a'_{[1](2)}) \cdot \phi(\gamma(m_{[1](2)})) \\ &= a'_{[0]} \otimes_A m_{[0]} \otimes [a'_{1} \cdot \phi(m_{1})] \otimes a'_{[1](2)} \cdot \phi(m_{[1](2)}) \\ &= (id_{F(M)} \otimes \Delta_{C'})\rho_{F(M)}(a' \otimes_A m), \end{aligned}$$

and

$$\begin{aligned} &(id_{F(M)} \otimes \varepsilon)\rho_{F(M)}(a' \otimes_A m) \\ &= (id_{F(M)} \otimes \varepsilon)(a'_{[0]} \otimes_A m_{[0]} \otimes a'_{[1]} \cdot \phi(m_{[1]})) \\ &= a'_{[0]}\varepsilon(a'_{[1]}) \otimes_A m_{[0]}\varepsilon(\phi(m_{[1]})) = a' \otimes_A m, \end{aligned}$$

as desired. And this completes the proof. \square

Theorem 4.2. Under the assumptions of Theorem 4.1, we have a functor $G : {}_{A'}\widetilde{\mathcal{H}}(\mathcal{M}_k)(H')^{C'} \rightarrow {}_A\mathcal{H}(\mathcal{M}_k)(H)^C$ which is right adjoint to F . G is defined by

$$G(M') = M' \square_{C'} C,$$

with structure maps

$$a \cdot (m' \otimes c) = \beta(a_{[0]}) \cdot m' \otimes a_{[1]} \cdot c, \quad (4.5)$$

$$\rho_{G(M')}(m' \otimes c) = \mu'^{-1}(m') \otimes c_{(1)} \otimes \gamma(c_{(2)}), \quad (4.6)$$

for all $a \in A$.

Proof. We first show that $G(M')$ is an object of ${}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$. It is not hard to check that $G(M')$ is a left (A, β) -module. Now we check that $G(M')$ is a right (C, γ) -comodule and satisfy the compatible condition. For any $m' \in M'$ and $a \in A, c \in C$, we have

$$\begin{aligned}\rho_{G(M')}(a \cdot (m' \otimes c)) &= \rho_{G(M')}(\beta(a_{[0]}) \cdot m' \otimes a_{[1]} \cdot c) \\ &= a_{[0]} \cdot \mu'^{-1}(m') \otimes a_{1} \cdot c_{(1)} \otimes \alpha(a_{[1](2)}) \cdot \gamma(c_{(2)}) \\ &= \beta(a_{[0][0]}) \cdot \mu'^{-1}(m') \otimes a_{[0][1]} \cdot c_{(1)} \otimes a_{[1]} \cdot \gamma(c_{(2)}) \\ &= a \cdot (\mu'^{-1}(m') \otimes c_{(1)} \otimes \gamma(c_{(2)})) = b' \rho_{G(M')}(m' \otimes c),\end{aligned}$$

i.e., the compatible condition holds. It remains to prove that $M' \square_{C'} C$ is a right (C, γ) -comodule. For any $m' \in M'$ and $a \in A$, we have

$$\begin{aligned}(\rho_{G(M')} \otimes id_{C'}) \rho_{G(M')}(m' \otimes_A c) &= (\rho_{G(M')} \otimes id_{C'})(\mu'^{-1}(m') \otimes c_{(1)} \otimes \gamma(c_{(2)})) \\ &= \mu'^{-2}(m') \otimes c_{(1)(1)} \otimes \gamma(c_{(1)(2)}) \otimes \gamma(c_{(2)}) \\ &= \mu'^{-2}(m') \otimes \gamma^{-1}(c_{(1)}) \otimes \gamma(c_{(2)(1)}) \otimes \gamma^2(c_{(2)(2)}) \\ &= \mu'^{-1}(m') \otimes c_{(1)} \otimes [\gamma(c_{(2)(1)}) \otimes \gamma(c_{(2)(2)})] \\ &= (id_{G(M')} \otimes \Delta_C) \rho_{G(M')}(m' \otimes c),\end{aligned}$$

and

$$\begin{aligned}(id_{G(M')} \otimes \varepsilon) \rho_{G(M')}(m' \otimes c) &= (id_{G(M')} \otimes \varepsilon)(\mu'^{-1}(m') \otimes c_{(1)} \otimes \gamma(c_{(2)})) \\ &= \mu'^{-1}(m') \otimes c_{(1)} \varepsilon(c_{(2)}) \otimes 1_C = m' \otimes c,\end{aligned}$$

as required.

$G(M') \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$ and the functorial properties can be checked in a straightforward way. Finally, we show that G is a right adjoint to F . Take $(M, \mu) \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$, define $\eta_M : M \rightarrow GF(M) = (M \otimes_A A') \square_{C'} C$ as follows: for all $m \in M$,

$$\eta_M(m) = m_{[0]} \otimes_A 1_{A'} \otimes m_{[1]}.$$

It is easy to see that $\eta_M \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$. Take $(M', \mu') \in {}_{A'}\widetilde{\mathcal{H}}(\mathcal{M}_k)(H')^{C'}$, define $\delta_{M'} : FG(M') \rightarrow M'$, where

$$\delta_{M'}(m' \otimes c) \otimes_A a' = \varepsilon_C(c) m' \cdot a',$$

It is easy to check that $\delta_{M'}$ is (A, β) -linear and therefore $\delta_{M'} \in {}_{A'}\widetilde{\mathcal{H}}(\mathcal{M}_k)(H')^{C'}$. We can also verify η and δ defined above are all natural transformations and satisfy

$$G(\delta_{M'}) \circ \eta_{G(M')} = I, \quad \delta_{F(M)} \circ F(\eta_M) = I,$$

for all $M \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$ and $M' \in {}_{A'}\widetilde{\mathcal{H}}(\mathcal{M}_k)(H')^{C'}$. And this completes the proof. \square

A morphism $\xi = (\varphi, \psi, \phi)$ between two monoidal Doi Hom-Hopf datum is called *monoidal* if the φ and ϕ are monoidal Hom-bialgebra maps. We now consider the particular situation where $H = H'$ and $A = A'$. The following result is a generalization of [3].

Theorem 4.3. Let $\xi = (id_H, id_A, \phi) : (H, A, C) \rightarrow (H, A, C')$ be a monoidal morphism of monoidal Doi Hom-Hopf datum. Then

$$G(C') = C. \quad (4.7)$$

Let $(M, \mu) \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$ be flat as a k -module, and take $(N, \nu) \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^{C'}$. If (C, γ) is a monoidal Hom-Hopf algebra, then

$$M \otimes G(N) \cong G(F(M) \otimes N) \text{ in } {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C. \quad (4.8)$$

If (C, γ) has a twisted antipode \overline{S} , then

$$G(N) \otimes M \cong G(N \otimes F(M)) \text{ in } {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C. \quad (4.9)$$

Proof. We know that $\varepsilon_{C'} \otimes id_C : C' \square_C C \rightarrow C$ is an isomorphism; the inverse map is $(\phi \otimes id_C) \Delta_C : C \rightarrow C' \square_C C$. It is clear that $\varepsilon_{C'} \otimes id_C$ is (A, β) -linear and (C, γ) -colinear. and this prove Eq.(4.7).

Now we define the map

$$\Gamma : M \otimes G(N) = M \otimes (N \square_{C'} C), G(F(M) \otimes N) = (F(M) \otimes N) \square_{C'} C,$$

which is given by

$$\Gamma(m \otimes (n_i \otimes c_i)) = (m_{[0]} \otimes n_i) \otimes m_{[1]} c_i.$$

Recall that $F(M) = M$ as an (A, β) -module, with (C', γ') -coaction given by

$$\rho_{F(M)}(m) = m_{[0]} \otimes \phi(m_{[1]}).$$

(1) Γ is well-defined, we have to show that

$$\Gamma(m_i \otimes (n_i \otimes c_i)) \in (F(M) \otimes N) \square_{C'} C.$$

This may be seen as follows: for any $m \in M$ and $n_i \square_{C'} c \in N \square_{C'} C$, we have

$$\begin{aligned} & (\rho_{F(M) \otimes N} \otimes id_C)((m_{[0]} \otimes n_i) \otimes m_{[1]} c_i) \\ &= (m_{[0][0]} \otimes n_{i[0]}) \otimes \phi(m_{[0][1]}) n_{i[1]} \otimes m_{[1]} c_i \\ &= (\mu(m_{[0]}) \otimes \nu(n_i)) \otimes \phi(m_{[0][1]}) \phi(c_{i(1)}) \otimes \gamma^{-1}(m_{[1]} c_{i(2)}) \\ &= (m_{[0]} \otimes n_i) \otimes [\phi(m_{[0][1]}) \phi(c_{i(1)}) \otimes m_{[1]} c_{i(2)}] \\ &= (id_{F(M) \otimes N} \otimes \rho_{C'})((m_{[0]} \otimes n_i) \otimes m_{[1]} c_i). \end{aligned}$$

(2) Γ is (A, β) -linear. Indeed, for any $a \in A, m \in M$ and $n_i \square_{C'} c \in N \square_{C'} C$, we have

$$\begin{aligned} & \Gamma(a \cdot (m \otimes (n_i \otimes c_i))) \\ &= \Gamma(a_{(1)} \cdot m \otimes (a_{(2)[0]} \cdot n_i \otimes a_{(2)[1]} \cdot c_i)) \\ &= (a_{(1)[0]} \cdot m_{[0]} \otimes a_{(2)[0]} \cdot n_i) \otimes (a_{(1)[1]} \cdot m_{[1]}) (a_{(2)[1]} \cdot c_i) \\ &= (a_{[0](1)} \cdot m_{[0]} \otimes a_{[0](2)} \cdot n_i) \otimes a_{(1)} \cdot (m_{[1]} c_i) \\ &= a_{[0]} \cdot (m_{[0]} \otimes n_i) \otimes a_{(1)} \cdot (m_{[1]} c_i) \\ &= a \cdot \Gamma(m \otimes (n_i \otimes c_i)). \end{aligned}$$

(3) Γ is (C, γ) -colinear. Indeed, for any $m \in M$ and $n_i \square_{C'} c \in N \square_{C'} C$, we have

$$\begin{aligned}
& \rho \circ \Gamma(m \otimes (n_i \otimes c_i)) \\
&= \rho((m_{[0]} \otimes n_i) \otimes m_{[1]} c_i) \\
&= (\mu^{-1}(m_{[0]}) \otimes \nu^{-1}(n_i)) \otimes m_{1} c_{i(1)} \otimes \gamma(m_{[1](2)} c_{i(2)}) \\
&= (m_{[0]} \otimes \nu^{-1}(n_i)) \otimes m_{[0][1]} c_{i(1)} \otimes m_{[1]} \gamma(c_{i(2)}) \\
&= (\Gamma \otimes id_C)(m_{[0]} \otimes (\nu^{-1}(n_i) \otimes c_{i(1)})) \otimes m_{[1]} \gamma(c_{i(2)}) \\
&= (\Gamma \otimes id_C) \circ \rho(m \otimes (n_i \otimes c_i)).
\end{aligned}$$

Assume (C, γ) has an antipode, define

$$\begin{aligned}
\Psi : (F(M) \otimes N) \square_{C'} C &\rightarrow M \otimes (N \square_{C'} C), \\
\Psi((m_i \otimes n_i) \otimes c_i) &= \mu^2(m_{i[0]}) \otimes (n_i \otimes S(m_{i[1]}) \gamma^{-2}(c_i)).
\end{aligned}$$

We have to show that Ψ is well-defined. (M, μ) is flat, so $M \otimes (N \square_{C'} C)$ is the equalizer of the maps

$$id_M \otimes id_N \otimes \rho_C : M \otimes N \otimes C \rightarrow M \otimes N \otimes C' \otimes C$$

and

$$id_M \otimes \rho_N \otimes id_C : M \otimes N \otimes C \rightarrow M \otimes N \otimes C' \otimes C.$$

Now take $(m_i \otimes n_i) \otimes c_i \in (F(M) \otimes N) \square_{C'} C$, then

$$(m_{i[0]} \otimes n_{i[0]}) \otimes \phi(m_{i[1]}) n_{i[1]} \otimes c_i = (\mu^{-1}(m_i) \otimes \nu^{-1}(n_i)) \otimes \phi(c_{i(1)}) \otimes \gamma(c_{i(2)}). \quad (4.10)$$

Therefore, we get

$$\begin{aligned}
& id_M \otimes id_N \otimes \rho_C(\mu^2(m_{i[0]}) \otimes (n_i \otimes S(m_{i[1]}) \gamma^{-2}(c_i))) \\
&= \mu^2(m_{i[0]}) \otimes (n_i \otimes \phi(S(m_{i[1](2)})) \gamma^{-2}(c_{i(1)})) \otimes S(m_{i1}) \gamma^{-2}(c_{i(2)}) \\
&= m_{i[0]} \otimes \nu^{-1}(n_i) \otimes \phi(S(\gamma(m_{i[1](2)})) \gamma^{-1}(c_{i(1)})) \otimes S(\gamma^2(m_{i1})) c_{i(2)}
\end{aligned}$$

and

$$\begin{aligned}
& id_M \otimes \rho_N \otimes id_C(\mu^2(m_{i[0]}) \otimes (n_i \otimes S(m_{i[1]}) \gamma^{-2}(c_i))) \\
&= \mu^2(m_{i[0]}) \otimes (n_{i[0]} \otimes n_{i[1]} \otimes S(m_{i[1]}) \gamma^{-2}(c_i)) \\
&= m_{i[0]} \otimes n_{i[0]} \otimes \gamma(n_{i[1]}) \otimes S(\gamma(m_{i[1]})) \gamma^{-1}(c_i).
\end{aligned}$$

Applying $(id_M \otimes \phi \otimes id_C) \circ (id_M \otimes (\Delta_C \circ S_C)) \circ \rho_M$ to the first factor of Eq.(4.10), we obtain

$$\begin{aligned}
& m_{i[0][0]} \otimes \phi(S(m_{i[0][1](2)})) \otimes S(m_{i[0]1}) \otimes n_{i[0]} \otimes \phi(m_{i[1]}) n_{i[1]} \otimes c_i \\
&= \mu^{-1}(m_{i[0]}) \otimes \phi(S(\gamma^{-1}(m_{i[1](2)}))) \otimes S(\gamma^{-1}(m_{i1})) \otimes \nu^{-1}(n_i) \otimes \phi(c_{i(1)}) \otimes \gamma(c_{i(2)}).
\end{aligned}$$

Applying $id_M \otimes \gamma^2 \otimes id_C \otimes id_N \otimes \gamma^{-1} \otimes \gamma^{-1}$ to the above identity, we have

$$\begin{aligned}
& m_{i[0][0]} \otimes \phi(S(\gamma^2(m_{i[0][1](2)}))) \otimes S(m_{i[0]1}) \otimes n_{i[0]} \otimes \gamma^{-1}(\phi(m_{i[1]}) n_{i[1]}) \otimes \gamma^{-1}(c_i) \\
&= \mu^{-1}(m_{i[0]}) \otimes \phi(S(\gamma(m_{i[1](2)}))) \otimes S(\gamma^{-1}(m_{i1})) \otimes \nu^{-1}(n_i) \otimes \phi(\gamma^{-1}(c_{i(1)})) \otimes c_{i(2)}.
\end{aligned}$$

Multiplying the second and the fifth factor, and also the third and sixth factor, we have

$$\begin{aligned} & \mu(m_{i[0]}) \otimes n_{i[0]} \otimes \gamma(n_{i[1]}) \otimes S(\gamma(m_{i[1]}))\gamma^{-1}(c_i) \\ = & \mu(m_{i[0]}) \otimes \nu^{-1}(n_i) \otimes \phi(S(\gamma(m_{i[1](2)}))\gamma^{-1}(c_{i(1)})) \otimes S(\gamma^2(m_{i1}))c_{i(2)}, \end{aligned}$$

and applying $\mu^{-1} \otimes id_N \otimes id_C \otimes id_C$ to the above identity, we obtain

$$\begin{aligned} & m_{i[0]} \otimes n_{i[0]} \otimes \gamma(n_{i[1]}) \otimes S(\gamma(m_{i[1]}))\gamma^{-1}(c_i) \\ = & m_{i[0]} \otimes \nu^{-1}(n_i) \otimes \phi(S(\gamma(m_{i[1](2)}))\gamma^{-1}(c_{i(1)})) \otimes S(\gamma^2(m_{i1}))c_{i(2)} \end{aligned}$$

or

$$id_M \otimes \rho_N \otimes id_C \circ (\Psi((m_i \otimes n_i) \otimes c_i)) = id_M \otimes id_N \otimes \rho_C \circ (\Psi((m_i \otimes n_i) \otimes c_i)).$$

Let us point out that Γ and Ψ are each other's inverses. In fact,

$$\begin{aligned} & \Gamma \circ \Psi((m_i \otimes n_i) \otimes c_i) \\ = & \Gamma(\mu^2(m_{i[0]}) \otimes (n_i \otimes S(m_{i[1]})\gamma^{-2}(c_i))) \\ = & (\mu^2(m_{i[0][0]}) \otimes n_i) \otimes \gamma^2(m_{i[0][1]})S(m_{i[1]})\gamma^{-2}(c_i) \\ = & (\mu^2(m_{i[0][0]}) \otimes n_i) \otimes [\gamma(m_{i[0][1]})S(m_{i[1]})]\gamma^{-1}(c_i) \\ = & (\mu(m_{i[0]}) \otimes n_i) \otimes [\gamma(m_{i1})S(\gamma(m_{i[1](2)}))]\gamma^{-1}(c_i) \\ = & (m_i \otimes n_i) \otimes c_i, \end{aligned}$$

and

$$\begin{aligned} & \Psi \circ \Gamma(m \otimes (n_i \otimes c_i)) \\ = & \Psi((m_{[0]} \otimes n_i) \otimes m_{[1]}c_i) \\ = & \mu^2(m_{[0][0]}) \otimes (n_i \otimes [S(\gamma^{-1}(m_{[0][1]}))\gamma^{-2}(m_{[1]})]\gamma^{-1}(c_i)) \\ = & \mu(m_{[0]}) \otimes (n_i \otimes [S(\gamma^{-1}(m_{1}))\gamma^{-1}(m_{[1](2)})]\gamma^{-1}(c_i)) \\ = & m \otimes (n_i \otimes c_i). \end{aligned}$$

The proof of Eq.4.9 is similar and left to the reader. \square

Corollary 4.4. Let (H, A, C) be a monoidal Doi Hom-Hopf Datum, $\Lambda: {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C \rightarrow {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)$ the functor forgetting the (C, γ) -coaction. For any flat Doi Hom-Hopf module (M, μ) , we have an isomorphism

$$M \otimes C \cong \Lambda(M) \otimes C$$

in ${}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$. If k is a field, then ${}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$ has enough injective objects, and any injective object in ${}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$ is a direct summand of an object of the form $I \otimes C$, where I is an injective (A, β) -module.

We have already proved that the category of Hom-Yetter-Drinfeld modules may be viewed as the category of Doi Hom-Hopf modules corresponding to a monoidal Doi Hom-Hopf Datum. Then we have the following corollary.

Corollary 4.5. Let (H, α) be a monoidal Hom-Hopf algebra with the bijective antipode. Then the category of Hom-Yetter-Drinfeld modules over (H, α) is a Grothendieck category with enough injective objects.

We continue with the dual version of Theorem 4.3.

Theorem 4.6. Let $\chi = (id_H, \psi, id_C) : (H, A, C) \rightarrow (H, A', C)$ be a monoidal morphism of monoidal Doi Hom-Hopf data. Then

$$F(A) = A'. \quad (4.11)$$

Let $(M, \mu) \in {}_A \widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$ be flat as a k -module, and take $(N, \nu) \in {}_{A'} \widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$. If (A', β') is a monoidal Hom-Hopf algebra, then

$$F(M) \otimes N \cong F(M \otimes G(N)) \text{ in } {}_A \widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C. \quad (4.12)$$

If (A', β') has a twisted antipode \overline{S} , then

$$N \otimes F(M) \cong F(G(N) \otimes M) \text{ in } {}_A \widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C. \quad (4.13)$$

Proof. We only prove Eq.(4.12) and similar for Eq.(4.11) and Eq.(4.13). Assume that (A', β') is a monoidal Hom-Hopf algebra and define

$$\Gamma : F(M \otimes G(N)) = A' \otimes_A M \otimes G(N) \rightarrow F(M) \otimes N = (A' \otimes_A M) \otimes N$$

by

$$\Gamma(a' \otimes (m \otimes n)) = (a'_{(1)} \otimes m) \otimes a'_{(2)} \cdot n,$$

for all $a' \in A', m \in M$ and $n \in N$. Γ is well-defined since

$$\begin{aligned} \Gamma(a' \psi(a) \otimes (m \otimes n)) &= (a'_{(1)} \psi(a_{(1)}) \otimes m) \otimes a'_{(2)} \psi(a_{(2)}) \cdot n \\ &= (a'_{(1)} \otimes a_{(1)} \cdot m) \otimes a'_{(2)} \psi(a_{(2)}) \cdot n \\ &= \Gamma(a' \otimes (a_{(1)} \cdot m \otimes \psi(a_{(2)}) \cdot n)) \\ &= \Gamma(a' \otimes a \cdot (m \otimes n)). \end{aligned}$$

It is easy to check that Γ is (A', β') -linear. Now we shall verify that Γ is (C, γ) -colinear based on Eq.(3.1). For any $a' \in A', m \in M$ and $n \in N$, we have

$$\begin{aligned} \rho(\Gamma(a' \otimes (m \otimes n))) &= \rho((a'_{(1)} \otimes m) \otimes a'_{(2)} \cdot n) \\ &= (a'_{(1)[0]} \otimes m_{[0]}) \otimes (a'_{(2)[0]} \cdot n_{[0]}) \otimes (a'_{(1)[1]} \otimes m_{[1]}) (a'_{(2)[1]} \cdot n_{[1]}) \\ &\stackrel{(3.1)}{=} (a'_{[0](1)} \otimes m_{[0]}) \otimes (a'_{[0](2)} \cdot n_{[0]}) \otimes a'_{[1]}(m_{[1]} n_{[1]}) \\ &= (\Gamma \otimes id_c)(a'_{[0]} \otimes (m_{[0]} \otimes n_{[0]})) \otimes a'_{[1]}(m_{[1]} n_{[1]}) \\ &= (\Gamma \otimes id_c) \rho(a' \otimes (m \otimes n)). \end{aligned}$$

The inverse of Γ is given by

$$\Psi((a' \otimes m) \otimes n) = \beta'^2(a'_{(1)}) \otimes (m \otimes S(a'_{(2)}) \nu^{-2}(n))$$

for all $a' \in A', m \in M$ and $n \in N$. One can check that Ψ is well-defined similar to Γ . Finally, we have

$$\begin{aligned} \Psi(\Gamma(a' \otimes (m \otimes n))) &= \Psi((a'_{(1)} \otimes m) \otimes a'_{(2)} \cdot n) \\ &= \beta'^2(a'_{(1)(1)}) \otimes (m \otimes S(a'_{(1)(2)}) \nu^{-2}(a'_{(2)} \cdot n)) \\ &= \beta'(a'_{(1)}) \otimes (m \otimes [S(\beta'^{-1}(a'_{(2)(1)})) \beta'^{-1}(a'_{(2)(2)})] \cdot \nu^{-1}(n)) \\ &= a' \otimes a' \otimes (m \otimes n) \end{aligned}$$

and

$$\begin{aligned}
\Gamma(\Psi((a' \otimes m) \otimes n)) &= \Gamma(\beta'^2(a'_{(1)}) \otimes (m \otimes S(a'_{(2)})\nu^{-2}(n))) \\
&= (\beta'^2(a'_{(1)(1)}) \otimes m) \otimes a'_{(2)} \cdot \beta'^2(a'_{(1)(2)}) \cdot S(a'_{(2)})\nu^{-2}(n) \\
&= (\beta'(a'_{(1)}) \otimes m) \otimes a'_{(2)} \cdot [\beta'(a'_{(2)(1)}) \cdot S(\beta'(a'_{(2)}))] \nu^{-1}(n) \\
&= (a' \otimes m) \otimes n,
\end{aligned}$$

as needed. The proof is completed. \square

5 Braidings on the category of Doi Hom-Hopf modules

Consider now a map $\mathcal{Q} : C \otimes C \rightarrow A \otimes A$, with twisted convolution inverse \mathcal{R} such that $(\beta \otimes \beta)\mathcal{Q} = \mathcal{Q}(\gamma \otimes \gamma)$ and $(\beta \otimes \beta)\mathcal{R} = \mathcal{R}(\gamma \otimes \gamma)$. This means that

$$\begin{aligned}
&\mathcal{R}(\mathcal{Q}^1(c_{(2)} \otimes d_{(2)})_{[1]} \cdot \gamma^{-1}(c_{(1)}) \otimes \mathcal{Q}^2(c_{(2)} \otimes d_{(2)})_{[1]} \cdot \gamma^{-1}(d_{(1)}))(\beta(\mathcal{Q}^2(c_{(2)} \otimes d_{(2)})_{[0]})) \\
&\quad \otimes \beta(\mathcal{Q}^1(c_{(2)} \otimes d_{(2)})_{[0]}) \\
&= \varepsilon_C(c)1_A \otimes \varepsilon_C(d)1_A,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{Q}(\mathcal{R}^2(c_{(2)} \otimes d_{(2)})_{[1]} \cdot \gamma^{-1}(c_{(1)}) \otimes \mathcal{R}^1(c_{(2)} \otimes d_{(2)})_{[1]} \cdot \gamma^{-1}(d_{(1)}))(\beta(\mathcal{R}^2(c_{(2)} \otimes d_{(2)})_{[0]})) \\
&\quad \otimes \beta(\mathcal{R}^1(c_{(2)} \otimes d_{(2)})_{[0]}) \\
&= \varepsilon_C(c)1_A \otimes \varepsilon_C(d)1_A,
\end{aligned}$$

for all $c, d \in C$. Sometimes, we write $\mathcal{Q}(c \otimes d) := \mathcal{Q}^1(c \otimes d) \otimes \mathcal{Q}^2(c \otimes d)$ for all $c, d \in C$.

Let $(M, \mu), (N, \nu) \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$. By Proposition 3.3 we know $(M \otimes N, \mu \otimes \nu) \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$. Define a map

$$\begin{aligned}
c_{M,N} : M \otimes N &\rightarrow N \otimes M, \\
c_{M,N}(m \otimes n) &= \mathcal{Q}(n_{[1]} \otimes m_{[1]})(n_{[0]} \otimes m_{[0]}).
\end{aligned} \tag{5. 1}$$

Next we will prove that $c_{M,N}$ is an isomorphism with an inverse map

$$\begin{aligned}
c_{M,N}^{-1} : N \otimes M &\rightarrow M \otimes N, \\
c_{M,N}^{-1}(n \otimes m) &= \mathcal{R}(n_{[1]} \otimes m_{[1]})(m_{[0]} \otimes n_{[0]}).
\end{aligned}$$

For any $m \in M$ and $n \in N$, we have

$$\begin{aligned}
&c_{M,N}^{-1} \circ c_{M,N}(m \otimes n) \\
&= c_{M,N}^{-1}(\mathcal{Q}(n_{[1]} \otimes m_{[1]})(n_{[0]} \otimes m_{[0]})) \\
&= \mathcal{R}((\mathcal{Q}^1(n_{[1]} \otimes m_{[1]}) \cdot n_{[0]})_{[1]} \otimes (\mathcal{Q}^2(n_{[1]} \otimes m_{[1]}) \cdot m_{[0]})_{[1]}) \\
&\quad ((\mathcal{Q}^2(n_{[1]} \otimes m_{[1]}) \cdot m_{[0]})_{[0]} \otimes (\mathcal{Q}^1(n_{[1]} \otimes m_{[1]}) \cdot n_{[0]})_{[0]}) \\
&= \mathcal{R}(\mathcal{Q}^1(\gamma(n_{[1](2)}) \otimes \gamma(m_{[1](2)}))_{[1]} \cdot n_{1} \otimes \mathcal{Q}^2(\gamma(n_{[1](2)}) \otimes \gamma(m_{[1](2)}))_{[1]} \cdot m_{1}) \\
&\quad (\mathcal{Q}^2(\gamma(n_{[1](2)}) \otimes \gamma(m_{[1](2)}))_{[0]} \cdot \mu^{-1}(m_{[0]}) \otimes \mathcal{Q}^1(\gamma(n_{[1](2)}) \otimes \gamma(m_{[1](2)}))_{[0]} \cdot \nu^{-1}(n_{[0]})) \\
&= (\mathcal{R}(\mathcal{Q}^1(n_{[1](2)} \otimes m_{[1](2)})_{[1]} \cdot \gamma^{-1}(n_{1}) \otimes \mathcal{Q}^2(n_{[1](2)} \otimes m_{[1](2)})_{[1]} \cdot \gamma^{-1}(m_{1})) \\
&\quad (\beta(\mathcal{Q}^2(n_{[1](2)} \otimes m_{[1](2)})_{[0]}) \otimes \beta(\mathcal{Q}^1(n_{[1](2)} \otimes m_{[1](2)})_{[0]})))(m_{[0]} \otimes n_{[0]})
\end{aligned}$$

$$\begin{aligned}
&= (\varepsilon_C(m_{[1]})1_A \otimes \varepsilon_C(n_{[1]})1_A)(m_{[0]} \otimes n_{[0]}) \\
&= m \otimes n.
\end{aligned}$$

So $c_{M,N}^{-1} \circ c_{M,N} = id_{M \otimes N}$. Similarly, we can prove $c_{M,N} \circ c_{M,N}^{-1} = id_{N \otimes M}$.

Our aim is now to give necessary and sufficient conditions on \mathcal{Q} such that the $c_{M,N}$ defines a braiding on the monoidal category of Doi Hom-Hopf modules. Recall from [14] that for any $(M, \mu), (N, \nu) \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$, the associativity and unit constraints are given by

$$\begin{aligned}
a_{M,N,P} : (M \otimes N) \otimes P &\rightarrow M \otimes (N \otimes P), \quad (m \otimes n) \otimes p \mapsto \mu(m) \otimes (n \otimes \pi^{-1}(p)), \\
l_M : k \otimes M &\rightarrow M, \quad k \otimes m \mapsto k\mu(m), \quad r_M : M \otimes k \rightarrow M, \quad m \otimes k \mapsto k\mu(m).
\end{aligned}$$

Next, we will find conditions under which $c_{M,N}$ is both an (A, β) -module map and a (C, γ) -comodule map, and satisfies the following conditions (for $P \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$)

$$a_{N,P,M} \circ c_{M,N \otimes P} \circ a_{M,N,P} = (id_N \otimes c_{M,P}) \circ a_{N,M,P} \circ (c_{M,N} \otimes id_P). \quad (5.2)$$

$$a_{N,P,M}^{-1} \circ c_{M \otimes N,P} \circ a_{M,N,P}^{-1} = (c_{M,P} \otimes id_N) \circ a_{M,P,N}^{-1} \circ (id_M \otimes c_{N,P}), \quad (5.3)$$

Recall from [12] that $A \otimes C$ can be made into a Doi Hom-Hopf module as follows: the (A, β) -action and (C, γ) -coaction on $A \otimes C$ are given by the formulas

$$\begin{cases} a \cdot (b \otimes c) = \beta^{-1}(a)b \otimes \gamma(c); \\ \rho_{A \otimes C}(b \otimes c) = (b_{[0]} \otimes c_{(1)}) \otimes b_{[1]}c_{(2)}, \end{cases}$$

for any $a, b \in A$ and $c \in C$.

In order to approach to our main result, we need some lemmas:

Lemma 5.1. Let $(M, \mu), (N, \nu) \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$. Then $c_{M,N}$ is (A, β) -linear if and only if the following condition is satisfied:

$$\mathcal{Q}(a_{(2)[1]} \cdot c \otimes a_{(1)[1]} \cdot d)(a_{(2)[0]} \otimes a_{(1)[0]}) = \Delta(a)\mathcal{Q}(c \otimes d) \quad (5.4)$$

for all $a \in A$ and $c, d \in C$.

Proof. If $c_{M,N}$ is (A, β) -linear then $a \triangleright c_{M,N}(m \otimes n) = c_{M,N}(a \triangleright (m \otimes n))$. We compute two sides of the equation as follows:

$$a \triangleright c_{M,N}(m \otimes n) = (a_{(1)} \otimes a_{(2)})\mathcal{Q}(n_{[1]} \otimes m_{[1]})(n_{[0]} \otimes m_{[0]})$$

and

$$\begin{aligned}
&c_{M,N}(a \triangleright (m \otimes n)) \\
&= \mathcal{Q}(a_{(2)[1]} \cdot n_{[1]} \otimes a_{(1)[1]} \cdot m_{[1]})(a_{(2)[0]} \cdot n_{[0]} \otimes a_{(1)[0]} \cdot m_{[0]}).
\end{aligned}$$

Conversely, considering these equations and taking $M = N = A \otimes C$ and $m = 1 \otimes c$ and $n = 1 \otimes d$ for all $c, d \in C$. Then we can get Eq.(5.4). \square

Definition 5.2. A *quasitriangular monoidal Hom-Hopf algebra* is a monoidal Hom-Hopf algebra (H, α) together with an invertible element $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$ such

that the following conditions hold:

$$\begin{aligned}
(QT1) \quad & \Delta(R^{(1)}) \otimes R^{(2)} = R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}, \\
(QT2) \quad & R^1 \otimes \Delta(R^2) = R^1 r^1 \otimes r^2 \otimes R^2, \\
(QT3) \quad & \varepsilon(R^{(1)}) R^{(2)} = 1_H, R^{(1)} \varepsilon(R^{(2)}) = 1_H, \\
(QT4) \quad & \Delta^{cop}(h) R = R \Delta(h), \\
(QT5) \quad & (\alpha \otimes \alpha)(R) = R,
\end{aligned}$$

where $\Delta^{cop}(h) = h_{(2)} \otimes h_{(1)}$ for all $h \in H$. (H, α) is called almost cocommutative if $\Delta^{cop}(h) R = R \Delta(h)$ holds.

Example 5.3. Suppose that $C = k$ and write $R = \mathcal{Q}(1 \otimes 1)$. Then Eq.(5.4) takes the form $R \Delta_A^{cop}(a) = \Delta_A(a) R$ and this means that (A, β) is almost cocommutative.

Lemma 5.4. Let $(M, \mu), (N, \nu) \in {}_A \widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$. Then $c_{M,N}$ is (C, γ) -colinear if and only if the following condition is satisfied:

$$\begin{aligned}
& \mathcal{Q}(d_{(2)} \otimes c_{(2)})_{[0]} \otimes m_C(\mathcal{Q}(d_{(2)} \otimes c_{(2)})_{[1]}(d_{(1)} \otimes c_{(1)})) \\
&= \mathcal{Q}(d_{(1)} \otimes c_{(1)}) \otimes c_{(2)} d_{(2)},
\end{aligned} \tag{5.5}$$

for all $c, d \in C$.

Proof. If $c_{M,N}$ is (C, γ) -colinear, then we do the following calculations:

$$\begin{aligned}
& \rho_{N \otimes M} c_{M,N}(m \otimes n) \\
&= \rho_{N \otimes M}(\mathcal{Q}(n_{[1]} \otimes m_{[1]})(n_{[0]} \otimes m_{[0]})) \\
&= \mathcal{Q}(n_{[1]} \otimes m_{[1]})_{[0]}(n_{[0][0]} \otimes m_{[0][0]}) \otimes m_C(\mathcal{Q}(n_{[1]} \otimes m_{[1]})_{[1]}(n_{[0][1]} \otimes m_{[0][1]})) \\
&= \mathcal{Q}(\gamma^{-1}(n_{[1](2)}) \otimes \gamma^{-1}(m_{[1](2)}))_{[0]}(\nu(n_{[0]}) \otimes \mu(m_{[0]})) \otimes m_C(\mathcal{Q}(\gamma^{-1}(n_{[1](2)}) \\
&\quad \otimes \gamma^{-1}(m_{[1](2)}))_{[1]}(n_{1} \otimes m_{[1](2)})).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(c_{M,N} \otimes id_C) \rho_{M \otimes N}(m \otimes n) &= \mathcal{Q}(n_{[0][1]} \otimes m_{[0][1]})(n_{[0][0]} \otimes m_{[0][0]}) \otimes (m_{[1]} n_{[1]}) \\
&= \mathcal{Q}(n_{1} \otimes m_{1})(\nu(n_{[0]}) \otimes \mu(m_{[0]})) \otimes \gamma^{-1}(m_{[1](2)} n_{[1](2)}).
\end{aligned}$$

Conversely, let $M = N = A \otimes C$ and take $m = 1 \otimes c$ and $n = 1 \otimes d$ for all $c, d \in C$. Then we can get Eq.(5.5). \square

Dual to quasitriangular monoidal Hom-Hopf algebras, a *coquasitriangular monoidal Hom-Hopf algebra* is a monoidal Hom-Hopf algebra (H, α) together with a bilinear form σ on (H, α) (i.e. $\sigma \in \text{Hom}(H \otimes H, k)$) such that the following axioms hold:

$$\begin{aligned}
(BR1) \quad & \sigma(hg, l) = \sigma(h, l_{(2)}) \sigma(g, l_{(1)}), \\
(BR2) \quad & \sigma(h, gl) = \sigma(h_{(1)}, g) \sigma(h_{(2)}, l), \\
(BR3) \quad & \sigma(h_{(1)}, g_{(1)}) g_{(2)} h_{(2)} = h_{(1)} g_{(1)} \sigma(h_{(2)}, g_{(2)}), \\
(BR4) \quad & \sigma(1_H, h) = \sigma(h, 1_H) = \varepsilon(h), \\
(BR5) \quad & \sigma(\alpha(h), \alpha(g)) = \sigma(h, g),
\end{aligned}$$

for all $h, g, l \in H$. (H, α) is called almost commutative if $\sigma(h_{(1)}, g_{(1)}) g_{(2)} h_{(2)} = h_{(1)} g_{(1)} \sigma(h_{(2)}, g_{(2)})$ holds.

Example 5.5. Suppose $A = k$. Then Eq.(5.5) takes the form

$$\mathcal{Q}(h_{(1)}, g_{(1)})g_{(2)}h_{(2)} = h_{(1)}g_{(1)}\mathcal{Q}(h_{(2)}, g_{(2)})$$

and this means that (A, β) almost commutative.

Lemma 5.6. Let $(M, \mu), (N, \nu), (P, \pi) \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$. Then Eq.(5.2) holds if and only if the following condition is satisfied, with $\mathcal{U} = \mathcal{Q}$:

$$\begin{aligned} & \mathcal{Q}^1(e \otimes \gamma(d_{(2)})) \otimes (\mathcal{U}^1(\gamma^{-1}(c) \otimes \mathcal{Q}^2(e \otimes \gamma(d_{(2)})))_{[1]}d_{(1)}) \\ & \otimes \mathcal{U}^2(\gamma^{-2}(c) \otimes \mathcal{Q}^2(e \otimes c_{(2)}))_{[1]}\gamma^{-1}(c_{(1)}) \mathcal{Q}^2(e \otimes \gamma(d_{(2)}))_{[0]} \\ = & \mathcal{Q}^1(e\gamma^{-1}(c) \otimes \gamma(m_{[1]}))_{(1)} \otimes \mathcal{Q}^1(e\gamma^{-1}(c) \otimes \gamma(c))_{(2)} \otimes \mathcal{Q}^2(\gamma^{-1}(e)\gamma^{-2}(c) \otimes d) \end{aligned} \quad (5.6)$$

for all $c, d, e \in C$.

Proof. If Eq.(5.2) holds, then we compute as follows:

$$\begin{aligned} & (id_N \otimes c_{M,P}) \circ a_{N,M,P} \circ (c_{M,N} \otimes id_P)((m \otimes n) \otimes p) \\ = & (id_N \otimes c_{M,P}) \circ a_{N,M,P}(\mathcal{Q}^1(n_{[1]} \otimes m_{[1]})n_{[0]} \otimes \mathcal{Q}^2(n_{[1]} \otimes m_{[1]})m_{[0]} \otimes p) \\ = & (id_N \otimes c_{M,P})(\beta(\mathcal{Q}^1(n_{[1]} \otimes m_{[1]}))\nu(n_{[0]}) \otimes (\mathcal{Q}^2(n_{[1]} \otimes m_{[1]})m_{[0]} \otimes \pi^{-1}(p))) \\ = & \beta(\mathcal{Q}^1(n_{[1]} \otimes m_{[1]}))\nu(n_{[0]}) \otimes \mathcal{U}(\gamma^{-1}(p_{[1]}) \otimes \mathcal{Q}^2(n_{[1]} \otimes m_{[1]})_{[1]}m_{[0][1]}) \\ & (\pi^{-1}(p_{[0]}) \otimes \mathcal{Q}^2(n_{[1]} \otimes m_{[1]})_{[0]}m_{[0][0]}) \\ = & \beta(\mathcal{Q}^1(n_{[1]} \otimes \gamma(m_{[1](2)})))\nu(n_{[0]}) \otimes \mathcal{U}(\gamma^{-1}(p_{[1]}) \otimes \mathcal{Q}^2(n_{[1]} \otimes \gamma(m_{[1](2)})))_{[1]}m_{1} \\ & (\pi^{-1}(p_{[0]}) \otimes \mathcal{Q}^2(n_{[1]} \otimes \gamma(m_{[1](2)})))_{[0]}\mu^{-1}(m_{[0]}) \\ = & \beta(\mathcal{Q}^1(n_{[1]} \otimes \gamma(m_{[1](2)})))\nu(n_{[0]}) \otimes (\mathcal{U}^1(\gamma^{-1}(p_{[1]}) \otimes \mathcal{Q}^2(n_{[1]} \otimes \gamma(m_{[1](2)})))_{[1]}m_{1} \\ & \pi^{-1}(p_{[0]}) \otimes \beta^{-1}(\mathcal{U}^2(\gamma^{-1}(p_{[1]}) \otimes \mathcal{Q}^2(n_{[1]} \otimes \gamma(m_{[1](2)})))_{[1]}m_{1})) \\ & \mathcal{Q}^2(n_{[1]} \otimes \gamma(m_{[1](2)}))_{[0]}m_{[0]}. \end{aligned}$$

Also we have

$$\begin{aligned} & a_{N,P,M} \circ c_{M,N \otimes P} \circ a_{M,N,P}((m \otimes n) \otimes p) \\ = & a_{N,P,M} \circ c_{M,N \otimes P}(\mu(m) \otimes (n \otimes \pi^{-1}(p))) \\ = & a_{N,P,M}((\Delta_A \otimes id_A)(\mathcal{Q}(n_{[1]}\gamma^{-1}(p_{[1]}) \otimes \gamma(m_{[1]})))((n_{[0]} \otimes \pi^{-1}(p_{[0]})) \otimes \mu(m_{[0]}))) \\ = & \beta(\mathcal{Q}^1(n_{[1]}\gamma^{-1}(p_{[1]}) \otimes \gamma(m_{[1]}))_{(1)})\nu(n_{[0]}) \otimes \mathcal{Q}^1(n_{[1]}\gamma^{-1}(p_{[1]}) \otimes \gamma(m_{[1]}))_{(2)}\pi^{-1}(p_{[0]}) \\ & \otimes \beta^{-1}(\mathcal{Q}^2(n_{[1]}\gamma^{-1}(p_{[1]}) \otimes \gamma(m_{[1]})))m_{[0]}. \end{aligned}$$

Conversely, take $M = N = P = A \otimes C$ and $m = 1 \otimes d$, and $n = 1 \otimes e$, and $p = 1 \otimes c$ for all $c, d, e \in C$. Then we obtain Eq.(5.6). \square

The proof of the following lemma is similar to that of the above lemma.

Lemma 5.7. Let $(M, \mu), (N, \nu), (P, \pi) \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$. Then Eq.(5.3) holds if and only if the following condition is satisfied, with $\mathcal{U} = \mathcal{Q}$:

$$\begin{aligned} & \mathcal{U}^1(\mathcal{Q}^1(c_{(2)} \otimes \gamma^{-1}(e))_{[1]}\gamma^{-1}(c_{(1)}) \otimes \gamma^{-2}(d))\mathcal{Q}^1(\gamma(c_{(2)}) \otimes e)_{[0]} \\ & \otimes \mathcal{U}^2(\mathcal{Q}^1(\gamma(c_{(2)}) \otimes e)_{[1]}c_{(1)} \otimes \gamma^{-1}(d)) \otimes \mathcal{Q}^2(c \otimes e) \\ = & \mathcal{Q}^1(c \otimes \gamma^{-2}(d)\gamma^{-1}(e)) \otimes \mathcal{Q}^2(\gamma(c) \otimes \gamma^{-1}(d)e)_{(1)} \otimes \mathcal{Q}^2(\gamma(c) \otimes \gamma^{-1}(d)e)_{(2)} \end{aligned} \quad (5.7)$$

for all $c, d, e \in C$.

Proof. If Eq.(5.3) holds, it follows that

$$\begin{aligned}
& (c_{M,P} \otimes id_N) \circ a_{M,P,N}^{-1} \circ (id_M \otimes c_{N,P})(m \otimes (n \otimes p)) \\
&= (c_{M,P} \otimes id_N) \circ a_{M,P,N}^{-1}(m \otimes \mathcal{Q}(p_{[1]} \otimes n_{[1]})(p_{[0]} \otimes n_{[0]})) \\
&= (c_{M,P} \otimes id_N)((\mu^{-1}(m) \otimes \mathcal{Q}^1(p_{[1]} \otimes n_{[1]})p_{[0]}) \otimes \beta(\mathcal{Q}^2(p_{[1]} \otimes n_{[1]}))\nu(n_{[0]})) \\
&= \mathcal{U}(\mathcal{Q}^1(p_{[1]} \otimes n_{[1]})_{[1]}p_{[0][1]} \otimes \gamma^{-1}(m_{[1]}))(\mathcal{Q}^1(p_{[1]} \otimes n_{[1]})_{[0]}p_{[0][0]} \otimes \mu^{-1}(m_{[0]})) \\
&\quad \otimes \beta(\mathcal{Q}^2(p_{[1]} \otimes n_{[1]}))\nu(n_{[0]}) \\
&= \{\beta^{-1}(\mathcal{U}^1(\mathcal{Q}^1(\gamma(p_{[1](2)}) \otimes n_{[1]})_{[1]}p_{1} \otimes \gamma^{-1}(m_{[1]})))\mathcal{Q}^1(\gamma(p_{[1](2)}) \otimes n_{[1]})_{[0]}\}p_{[0]} \\
&\quad \otimes \mathcal{U}^2(\mathcal{Q}^1(\gamma(p_{[1](2)}) \otimes n_{[1]})_{[1]}p_{1} \otimes \gamma^{-1}(m_{[1]}))\mu^{-1}(m_{[0]}) \otimes \beta(\mathcal{Q}^2(p_{[1]} \otimes n_{[1]}))\nu(n_{[0]})
\end{aligned}$$

and

$$\begin{aligned}
& a_{P,M,N}^{-1} \circ c_{M \otimes N, P} \circ a_{M,N,P}^{-1}(m \otimes (n \otimes p)) \\
&= a_{P,M,N}^{-1} \circ c_{M \otimes N, P}((\mu^{-1}(m) \otimes n) \otimes \pi(p)) \\
&= a_{P,M,N}^{-1} \mathcal{Q}(\gamma(p_{[1]}) \otimes \gamma^{-1}(m_{[1]})n_{[1]})(\pi(p_{[0]}) \otimes (\mu^{-1}(m_{[0]}) \otimes n_{[0]})) \\
&= \beta^{-1}(\mathcal{Q}^1(\gamma(p_{[1]}) \otimes \gamma^{-1}(m_{[1]})n_{[1]}))p_{[0]} \otimes \mathcal{Q}^2(\gamma(p_{[1]}) \otimes \gamma^{-1}(m_{[1]})n_{[1]})_{(1)}\mu^{-1}(m_{[0]}) \\
&\quad \otimes \beta(\mathcal{Q}^2(\gamma(p_{[1]}) \otimes \gamma^{-1}(m_{[1]})n_{[1]}))_{(2)}\nu(n_{[0]}).
\end{aligned}$$

Conversely, take $M = N = P = A \otimes C$ and $m = 1 \otimes d$, and $n = 1 \otimes e$, and $p = 1 \otimes c$ for all $c, d, e \in C$. Then we obtain Eq.(5.7). \square

Therefore, we can summarize our results as follows.

Theorem 5.8. Let (H, A, C) be a monoidal Doi Hom-Hopf datum, and $\mathcal{Q} : C \otimes C \longrightarrow A \otimes A$ a twisted convolution invertible map. For $(M, \mu), (N, \nu) \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$, then the family of maps

$$c_{M,N} : M \otimes N \rightarrow N \otimes M, \quad c_{M,N}(m \otimes n) = \mathcal{Q}(n_{[1]} \otimes m_{[1]})(n_{[0]} \otimes m_{[0]})$$

defines a braiding on the category of Doi Hom-Hopf modules ${}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$ if and only if Eqs.(5.4), (5.5), (5.6), (5.7) are satisfied.

Example 5.9. (1) Take $A = k$ and write

$$R = \mathcal{Q}(1_C \otimes 1_C) = \sum R^{(1)} \otimes R^{(2)} = \sum r^{(1)} \otimes r^{(2)}.$$

Eqs.(5.6) and (5.7) take the form

$$\begin{aligned}
\Delta(R^{(1)}) \otimes R^{(2)} &= R^{(1)} \otimes r^{(1)} \otimes r^{(2)} R^{(2)}, \\
R^{(1)} \otimes \Delta(R^{(2)}) &= r^{(1)} R^{(1)} \otimes r^{(2)} \otimes R^{(2)}.
\end{aligned}$$

and the braiding is

$$c_{M,N} : M \otimes N \rightarrow N \otimes M, \quad c_{M,N}(m \otimes n) = R^{(2)} \cdot \nu^{-1}(n) \otimes R^{(1)} \cdot \mu^{-1}(m).$$

Assume that R is α -invariant (i.e. $\alpha(R^{(1)}) \otimes \alpha(R^{(2)}) = R^{(1)} \otimes R^{(2)}$). We conclude that the conditions of Theorem 5.8 are satisfied if and only if (C, R^{-1}) is a quasitriangular monoidal Hom-bialgebra.

(2) If $C = k$, then Eqs.(5.6) and (5.7) take the form

$$\begin{aligned}\sigma(hg, l) &= \sigma(h, l_{(1)})\sigma(g, l_{(2)}), \\ \sigma(h, gl) &= \sigma(h_{(1)}, l)\sigma(h_{(2)}, g).\end{aligned}$$

and the braiding is

$$\begin{aligned}c_{M,N} : M \otimes N &\rightarrow N \otimes M, \\ c_{M,N}(m \otimes n) &= \sigma(n_{[1]}, m_{[1]})\nu(n_{[0]}) \otimes \mu(m_{[0]}).\end{aligned}$$

Assume that σ is α -invariant (i.e. $\sigma(\alpha(h), \alpha(g)) = \sigma(h, g)$ for all $h, g \in H$) and we conclude that the conditions of Theorem 5.8 are satisfied if and only if (A, σ) is a coquasitriangular monoidal Hom-bialgebra.

(3) Let (H, α) be a monoidal Hom-Hopf algebra with bijective antipode. We have seen that the category of Doi Hom-Hopf modules ${}_H\widetilde{\mathcal{H}}(\mathcal{M}_k)(H^{op} \otimes H)^H$ and the category of Hom-Yetter-Drinfeld modules ${}_H\mathcal{HYD}^H$ are isomorphic. Recall from [14] that ${}_H\mathcal{HYD}^H$ is a braided category. The braiding is induced by

$$c_{M,N} : M \otimes N \rightarrow N \otimes M, \quad m \otimes n \mapsto \nu(n_{[0]}) \otimes n_{[1]}\mu^{-1}(m).$$

The corresponding map \mathcal{Q} is

$$\mathcal{Q} : H \otimes H \rightarrow H \otimes H, \quad h \otimes k \mapsto \eta(\varepsilon(k)) \otimes h.$$

It is straightforward to check that \mathcal{Q} satisfies the conditions of Theorem 5.8. \square

6 The smash product of Hom-bialgebras and the Drinfel'd double

Let (A, β) be a right (H, α) -Hom comodule algebra, and (B, ζ) a left (H, α) -Hom module coalgebra. Consider the following version of smash product $A \# B$ with the multiplication given by

$$(a \# b)(c \# d) = a\beta(c_{[0]})\#(\zeta^{-1}(b) \leftarrow c_{[1]})d.$$

Then $A \# B$ is a Hom associative algebra with unit $1_A \# 1_B$.

If (C, γ) is a faithfully projective left (H, α) -Hom module coalgebra, Then (C^*, γ^*) is a right (H, α) -Hom module algebra. The right (H, α) -action is given by

$$(c^* \leftarrow h, c) = (c^*, h \cdot c).$$

Given $(M, \mu) \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$, we define an $A \# C^*$ -action on (M, μ) as follows

$$(a \# c^*) \cdot m = \langle c^*, m_{[1]} \rangle a \cdot m_{[0]}.$$

Assume that (A, β) and (B, ζ) are both monoidal Hom-bialgebras, consider $\Delta_{A \# B}$ and $\varepsilon_{A \# B}$ defined by

$$\begin{aligned}\Delta_{A \# B}(a \# b) &= (a_{(1)} \# b_{(1)}) \otimes (a_{(2)} \# b_{(2)}), \\ \varepsilon_{A \# B}(a \# b) &= \varepsilon_A(a)\varepsilon_B(b).\end{aligned}$$

Proposition 6.1. With notations as above. If

$$\begin{aligned} & \Delta_A(\beta(a_{[0]})) \otimes \Delta_A(\zeta^{-1}(b) \leftarrow a_{[1]}) \\ &= \beta(a_{(1)[0]}) \otimes \beta(a_{(2)[0]}) \otimes (\zeta^{-1}(b_{(1)}) \leftarrow a_{1}) \otimes (\zeta^{-1}(b_{(2)}) \leftarrow a_{[1](2)}) \end{aligned} \quad (6.1)$$

and

$$\varepsilon_A(a_{[0]}) \otimes \varepsilon_B(b \leftarrow a_{[1]}) = \varepsilon_A(a) \varepsilon_B(b), \quad (6.2)$$

for all $a \in A$ and $b \in B$, then $A \# B$ is a monoidal Hom-bialgebra. If (A, β) and (B, ζ) are both monoidal Hom-Hopf algebras, then $A \# B$ is a monoidal Hom-Hopf algebras with the antipode given by

$$S_{A \# B}(a \# b) = S(\beta(a))_{[0]} \# (S(\zeta^{-1}(b)) \leftarrow S(a)_{[1]}).$$

Proof. We leave it to the reader to show that $\Delta_{A \# B}$ is multiplicative if and only if Eq.(6.1) holds, and $\varepsilon_{A \# B}$ is multiplicative if and only if Eq.(6.2) holds. We show that the antipode defined above is convolution invertible. In fact, we have

$$\begin{aligned} & (a_{(1)} \# b_{(1)}) S_{A \# B}(a_{(2)} \# b_{(2)}) \\ &= (a_{(1)} \# b_{(1)}) (S(\beta(a_{(2)}))_{[0]} \otimes (S(\zeta^{-1}(b_{(2)})) \leftarrow S(a_{(2)})_{[1]})) \\ &= a_{(1)} S(\beta^2(a_{(2)}))_{[0][0]} \# (\zeta^{-1}(b_{(1)}) \leftarrow S(\beta(a_{(2)}))_{[0][1]}) (S(\zeta^{-1}(b_{(2)})) \leftarrow S(a_{(2)})_{[1]}) \\ &= a_{(1)} S(\beta(a_{(2)}))_{[0]} \# (\zeta^{-1}(b_{(1)}) \leftarrow S(\beta(a_{(2)}))_{1}) (S(\zeta^{-1}(b_{(2)})) \leftarrow S(\beta(a_{(2)}))_{[1](2)})) \\ &= a_{(1)} S(\beta(a_{(2)}))_{[0]} \# (\zeta^{-1}(b_{(1)}) S(\zeta^{-1}(b_{(2)}))) \leftarrow S(\beta(a_{(2)}))_{[1]} \\ &= \varepsilon_A(a) \varepsilon_B(b), \end{aligned}$$

and

$$\begin{aligned} & S_{A \# B}(a_{(1)} \# b_{(1)}) (a_{(2)} \# b_{(2)}) \\ &= (S(\beta(a_{(1)}))_{[0]} \otimes (S(\zeta^{-1}(b_{(1)})) \leftarrow S(a_{(1)})_{[1]})) (a_{(2)} \# b_{(2)}) \\ &= S(\beta(a_{(1)}))_{[0]} \beta(a_{(2)[0]}) \# (S(\zeta^{-1}(b_{(1)})) \leftarrow S(a_{(1)})_{[1]}) a_{(2)[1]} b_{(2)} \\ &= \varepsilon_A(a) \varepsilon_B(b), \end{aligned}$$

as desired. \square

Proposition 6.2. Let (H, A, C) be a monoidal Doi Hom-Hopf module. Assume (C, γ) is faithfully projective as a k -module. Then (A, β) and (C^*, γ^*) satisfy Eqs.(6.1), (6.2), ${}_A \widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$ and $A \# C^*$ -Hom modules as monoidal categories.

Proof. We leave the proof to the reader similar to [4]. \square

Example 6.3. Assume that (H, α) is faithfully projective as a k -module. The monoidal Hom-algebra $A \# C^*$ is nothing else than the Drinfel'd double $D(H) = H \# H^*$. Then we can give the multiplication as follows

$$(h \# f)(k \# g) = h \alpha^2(h_{(2)(1)}) \# \langle \alpha^{*-2}(f), \alpha(h_{(2)(2)}) \rightharpoonup \bullet \leftarrow S^{-1}(\alpha^{-1}(h)) \rangle g.$$

Now let (H, A, C) be a monoidal Doi Hom-Hopf datum, and $\mathcal{Q} : C \otimes C \longrightarrow A \otimes A$ a twisted convolution invertible map satisfying Eqs.(5.4), (5.5), (5.6), (5.7). \mathcal{Q} induces the map

$$\widetilde{\mathcal{Q}} : k \rightarrow (A \# C^*) \otimes (A \# C^*).$$

The braiding on ${}_A \widetilde{\mathcal{H}}(\mathcal{M}_k)(H)^C$ translates into a braiding on $A \# C^*$ -Hom modules. This means that $A \# C^*$ is a quasitriangular monoidal Hom-Hopf algebra. \square

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